CIRCUIT ANALYSIS TECHNIQUES AND NETWORK THEOREMS

CHAPTER - 8
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Standard Signs and Conventions

If the loop enters the negative sign and goes out on plus sign, the **battery emf is positive** ($+E$).
If the loop enters the positive sign and goes out on negative sign, the **battery emf is negative** ($-E$).

If the loop direction is the same as the current direction, the **resistance voltage drop is negative** ($-E_R$ or $-IR$).
If the loop direction is opposite to the current direction, the **resistance voltage drop is positive** ($+E_R$ or $+IR$).
Maxwell’s Loop (Mesh) Method

This method involves a set of independent loop currents assigned to as many meshes as it exist in the circuit and these currents are employed in connection with appropriate resistances when the KVL equations are written.
Maxwell’s Loop (Mesh) Method

The arbitrarily assumed loop currents may or may not exist in the various resistors but when determined will readily yield the desired values by simple algebraic additions.
The steps used in solving a circuit using mesh analysis are as follows:

1. Arbitrarily assign a clockwise current to each interior closed loop in the network. Although the assigned current may be in any direction, a clockwise direction is used to make later work simpler.
The steps used in solving a circuit using mesh analysis are as follows:

2. Using the assigned loop currents, indicate the voltage polarities across all resistors in the circuit. For a resistor which is common to two loops, the polarities of the voltage drop due to each loop current should be indicated on the appropriate side of the component.
The steps used in solving a circuit using mesh analysis are as follows:

3. Applying Kirchhoff’s voltage law, write the loop equations for each loop in the network. Do not forget that resistors which are common to two loops will have two voltage drops, one due to each loop.
The steps used in solving a circuit using mesh analysis are as follows:

4. Solve the resultant simultaneous linear equations.
5. Branch currents are determined by algebraically combining the loop currents which are common to the branch.
EXAMPLE 1. Find the current in each branch for the circuit of Figure.

Solution:
Step 1: Loop currents are assigned as shown. These currents are designated $I_1$ and $I_2$.
Step 2: Voltage polarities are assigned according to the loop currents. Notice that the resistor $R_2$ has two different voltage polarities due to the different loop currents.
Step 3: The loop equations are written by applying Kirchhoff’s voltage law in each of the loops. The equations are as follows:

Loop 1: \[ 6V - (2\Omega)I_1 - (2\Omega)I_1 + (2\Omega)I_2 - 4V = 0 \]

Loop 2: \[ 4V - (2\Omega)I_2 + (2\Omega)I_1 - (4\Omega)I_2 + 2V = 0 \]

Loop 1: \[ (4\Omega)I_1 - (2\Omega)I_2 = 2V \]

Loop 2: \[ -(2\Omega)I_1 + (6\Omega)I_2 = 6V \]
Using determinants, the loop equations are easily solved as:

\[
I_1 = \frac{\begin{vmatrix} 2 & -2 \\ 6 & 6 \\ 4 & -2 \\ -2 & 6 \end{vmatrix}}{24 - 4} = \frac{12 + 12}{20} = \frac{24}{20} = 1.20 \text{ A}
\]

and

\[
I_2 = \frac{\begin{vmatrix} 4 & 2 \\ -2 & 6 \\ 4 & -2 \\ -2 & 6 \end{vmatrix}}{24 - 4} = \frac{24 + 4}{20} = \frac{28}{20} = 1.40 \text{ A}
\]

The branch current for \( R_2 \) is found by combining the loop currents through this resistor:

\[
I_{R_2} = 1.40 \text{ A} - 1.20 \text{ A} = 0.20 \text{ A}
\]
Superposition Theorem

In a network of resistors, the current in any resistor is equal to the \textit{algebraic sum of the currents} delivered by each independent source assuming that each source is acting alone or independently with respect to the others.

The total current in any part of a linear circuit equals the algebraic sum of the currents produced by each source separately.
Superposition Theorem

REMEMBER:
If a source is operating alone, the other
*current sources are open circuited* while
the other *voltage sources are short
circuited.*
Superposition Theorem

The theorem states the following:

The total current through or voltage across a resistor or branch may be determined by summing the effects due to each independent source.
To apply the superposition theorem:

1. It is necessary to remove all sources other than the one being examined.

2. In order to “zero” a voltage source, we replace it with a short circuit, since the voltage across a short circuit is zero volts.

3. A current source is zeroed by replacing it with an open circuit, since the current through an open circuit is zero amps.
EXAMPLE 2. Consider the circuit of Figure below;

![Circuit Diagram]

a. Determine the current in the load resistor, $R_L$.
b. Verify that the superposition theorem does not apply to power.
Solution:
a. We first determine the current through $R_L$ due to the voltage source by removing the current source and replacing in with an open circuit (zero amps) as shown.

The resulting current through $R_L$ is determined from Ohm's law as:

$$I_{L(1)} = \frac{20 \, V}{16 \, \Omega + 24 \, \Omega} = 0.500 \, A$$
Next, determine the current through $R_L$ due to the current source by removing the voltage source and replacing it with a short circuit (zero volts) as shown.

The resulting current through $R_L$ is found with the current divider rule as:

$$I_{L(2)} = -\left(\frac{24\Omega}{24\Omega + 16\Omega}\right)(2A) = -1.20A$$
The resultant current through $R_L$ is found by applying the superposition theorem:

$$I_L = 0.5\, A - 1.2\, A = -0.70\, A$$

- The negative sign indicates that the current through $R_L$ is opposite to the assumed reference direction.
- Consequently, the current through $R_L$ will, in fact, be upward with a magnitude of 0.7 A.

b. If we assume (incorrectly) that the superposition theorem applies for power, we would have the power due the first source given as

$$P_1 = I_{L(1)}^2 R_L = (0.5\, A)^2\ (16\Omega) = 4.0\, W$$
and the power due the second source as

\[ P_2 = I_{L(2)}^2 R_L = (1.2 \ A)^2 (16\ \Omega) = 23.04\ W \]

The total power, if superposition applies, would be

\[ P_T = P_1 + P_2 = 4.0\ W + 23.04\ W = 27.04\ W \]

Clearly, this result is wrong, since the actual power dissipated by the load resistor is correctly given as

\[ P_L = I_L^2 R_L = (0.7 \ A)^2 (16\ \Omega) = 7.84\ W \]
For this procedure, every junction in the network that represents a connection of three or more branches is regarded as a node.
Moreover, considering one of the nodes as a reference or zero-potential point, current equations are then written for the remaining junctions; thus, a solution is possible with $n - 1$ equations, where $n$ is the number of nodes.
By KCL at node A:

\[ I_1 = I_2 + I_3 \]

\[ \frac{E_1 - V_A}{R_1} = \frac{V_A}{R_2} + \frac{V_A - V_B}{R_3} \]

\[ V_A \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) - \frac{E_1 - V_B}{R_1} = 0 \]

By KCL at node B:

\[ I_4 = I_3 + I_5 \]

\[ \frac{V_B}{R_4} = \frac{V_A - V_B}{R_3} + \frac{E_2 - V_B}{R_5} \]

\[ V_B \left( \frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5} \right) - \frac{E_2 - V_A}{R_5} = 0 \]
EXAMPLE 3. Apply nodal analysis to the network of Figure below.

\[ \text{Diagram:} \]
- \( R_1 = 6 \Omega \)
- \( R_2 = 12 \Omega \)
- \( E = 24 \text{ V} \)
- Current Source: 1A

\[ \text{Analysis:} \]

Apply KCL at the nodes to find the current through each resistor and the voltage drops across them.
Solution:
Steps 1 and 2: The network has two nodes, as shown. The lower node is defined as the reference node at ground potential (zero volts), and the other node as $V_1$, the voltage from node 1 to ground.
Step 3: $I_1$ and $I_2$ are defined as leaving the node in Figure below, and Kirchhoff’s current law is applied as follows:

$$I = I_1 + I_2$$

The current $I_2$ is related to the nodal voltage $V_1$ by Ohm’s law:

$$I_2 = \frac{V_{R_2}}{R_2} = \frac{V_1}{R_2}$$
The current $I_1$ is also determined by Ohm’s law as follows:

$$I_1 = \frac{V_{R_1}}{R_1}$$

with

$$V_{R_1} = V_1 - E$$
Substituting into the Kirchhoff’s current law equation:

\[ I = I_1 + I_2 = \frac{V_1 - E}{R_1} + \frac{V_1}{R_2} \]

and rearranging,

\[ I = \frac{V_1}{R_1} - \frac{E}{R_1} + \frac{V_1}{R_2} = V_1 \left( \frac{1}{R_1} + \frac{1}{R_2} \right) - \frac{E}{R_1} \]

Substituting numerical values,

\[ V_1 \left( \frac{1}{6\Omega} + \frac{1}{12\Omega} \right) = \frac{24V}{6\Omega} + 1A = 4A + 1A \]

\[ V_1 \left( \frac{1}{4\Omega} \right) = 5A \]

\[ V_1 = 20V \]
The currents \( I_1 \) and \( I_2 \) can then be determined using the preceding equations:

\[
I_1 = \frac{V_1 - E}{R_1} = \frac{20V - 24V}{6\Omega} = \frac{-4V}{6\Omega} = -0.667 \text{A}
\]

The minus sign indicates simply that the current \( I_1 \) has a direction opposite to that appearing in Figure above.

\[
I_2 = \frac{V_1}{R_2} = \frac{20V}{12\Omega} = 1.667 \text{A}
\]
THEVENIN’S THEOREM

- He is active in the study and design of telegraphic systems (including underground transmission), cylindrical condensers (capacitors), and electromagnetism.
- It appeared under the heading of “Sur un nouveau théorème d’électricité dynamique” (“On a new theorem of dynamic electricity”) and was originally referred to as the *equivalent generator theorem*. 
There is some evidence that a similar theorem was introduced by Hermann von Helmholtz in 1853. However, Professor Helmholtz applied the theorem to animal physiology and not to communication or generator systems, and therefore he has not received the credit in this field that he might deserve.

In the early 1920s AT&T did some pioneering work using the equivalent circuit and may have initiated the reference to the theorem as simply Thévenin’s theorem.
THEVENIN’S THEOREM

- In fact, Edward L. Norton, an engineer at AT&T at the time, introduced a current source equivalent of the Thévenin equivalent currently referred to as the Norton equivalent circuit.
- As an aside, Commandant Thévenin was an avid skier and in fact was commissioner of an international ski competition in Chamonix, France, in 1912.
Thévenin’s theorem states the following:

Any two-terminal, linear bilateral dc network can be replaced by an equivalent circuit consisting of a voltage source and a series resistor, as shown
THEVENIN’S THEOREM

Any combination of batteries and resistances with two terminals can be replaced by a single voltage source $V_{TH}$ and a single series resistor ($R_{TH}$).

The value of $V_{TH}$ is the open circuit voltage at the terminals, and the value of $R_{TH}$ is $V_{TH}$ divided by the current with the terminals short circuited.
SOLUTION 1:

1. Calculate the output voltage, \( V_{AB} \), when in open circuit condition (no load resistor - meaning infinite resistance). This is \( V_{Th} \).

2. Calculate the output current, \( I_{AB} \), when those leads are short circuited (load resistance is 0). \( R_{Th} \) equals \( V_{Th} \) divided by this \( I_{AB} \).
SOLUTION 2:

a. Now replace voltage sources with short circuits and current sources with open circuits.

b. Replace the load circuit with an imaginary ohm meter and measure the total resistance, $R$, "looking back" into the circuit. This is $R_{Th}$.

The equivalent circuit is a voltage source with voltage $V_{Th}$ in series with a resistance $R_{Th}$. 
Example 4: Determine the Thévenin equivalent circuit external to the resistor $R_L$ for the circuit of Figure below. Use the Thévenin equivalent circuit to calculate the current through $R_L$. 
**Solution:**

Step 1: Removing the load resistor from the circuit and labeling the remaining terminals, we obtain the circuit shown below.
Solution:
Step 2: Setting the sources to zero, we have the circuit below.
Solution:

Step 3: The Thévenin resistance between the terminals is $R_{Th} = 24 \, \Omega$.

Step 4: From figure above, the open-circuit voltage between terminals $a$ and $b$ is found as $V_{ab} = 20V - (24\Omega)(2A) = -28.0 \, V$

Step 5: The resulting Thévenin equivalent circuit is
Using this Thévenin equivalent circuit, we easily find the current through $R_L$ as

$$I_L = \left( \frac{28V}{24\Omega + 16\Omega} \right) = 0.70A$$
NORTON’S THEOREM

The theorem states the following:

“Any two-terminal linear bilateral dc network can be replaced by an equivalent circuit consisting of a current source and a parallel resistor”
The steps leading to the proper values of $I_N$ and $R_N$ are now listed:

1. Remove that portion of the network across which the Norton equivalent circuit is found.

2. Mark the terminals of the remaining two-terminal network.

3. Calculate $R_N$ by first setting all sources to zero (voltage sources are replaced with short circuits, and current sources with open circuits) and then finding the resultant resistance between the two marked terminals.
The steps leading to the proper values of $I_N$ and $R_N$ are now listed:

4. If the internal resistance of the voltage and/or current sources is included in the original network, it must remain when the sources are set to zero. Since $R_N = R_{Th}$, the procedure and value obtained using the approach described for Thévenin’s theorem will determine the proper value of $R_N$. 
The steps leading to the proper values of $I_N$ and $R_N$ are now listed:

5. Calculate $I_N$ by first returning all sources to their original position and then finding the short-circuit current between the marked terminals. It is the same current that would be measured by an ammeter placed between the marked terminals.
The steps leading to the proper values of $I_N$ and $R_N$ are now listed:

6. Draw the Norton equivalent circuit with the portion of the circuit previously removed replaced between the terminals of the equivalent circuit.
EXAMPLE 5. Find the Norton equivalent circuit for the network in the shaded area of Figure below.
Solution:
Steps 1 and 2 are shown in Figure below

Step 3. Shown in Figure below, and

\[
R_N = R_1 \parallel R_2 = 3\Omega \parallel 6\Omega \\
R_N = \frac{(3\Omega)(6\Omega)}{3\Omega + 6\Omega} = \frac{18\Omega}{9} = 2\Omega
\]
Step 4. The short-circuit connection between terminals a and b is in parallel with $R_2$ and eliminates its effect. $I_N$ is therefore the same as through $R_1$, and the full battery voltage appears across $R_1$;
Since;

\[ V_2 = I_2 R_2 = (0) 6 \Omega = 0V \]

Therefore,

\[ I_N = \frac{E}{R_1} = \frac{9V}{3\Omega} = 3A \]
Step 5: This circuit is the same as the first one considered in the development of Thévenin’s theorem. A simple conversion indicates that the Thévenin circuits are, in fact, the same

\[ R_{Th} = R_N = 2 \, \Omega \]

\[ E_{Th} = I_NR_N = (3 \, A)(2 \, \Omega) = 6 \, V \]
The transformation is used to establish equivalence for networks with 3 terminals. Where three elements terminate at one point (node) and none is a source, the node is eliminated by transforming the impedances.
Y-Δ TRANSFORMATION

The Y-Δ transform (also written Y-delta or Wye-delta), Kennelly's delta-star transformation, star-mesh transformation or T-Π (or T-pi) transform is a mathematical technique to simplify analysis of an electrical network.

The name derives from the shapes of the circuit diagrams, which look respectively like the letter Y and the Greek capital letter Δ.
Δ–Y TRANSFORMATION

Δ (Delta) Configuration

Y (Wye) Configuration

R_a, R_b, R_c, R_1, R_2, R_3
General Idea:

\[ R_Y = \frac{R_{\Delta_{\text{adjacent1}}} \cdot xR_{\Delta_{\text{adjacent2}}}}{\sum R_\Delta} \]

\[ R_1 = \frac{R_a R_b}{R_a + R_b + R_c} \]

\[ R_2 = \frac{R_b R_c}{R_a + R_b + R_c} \]

\[ R_3 = \frac{R_a R_c}{R_a + R_b + R_c} \]

Balanced System:

\[ R_\Delta = 3R_Y \]
EXAMPLE 6. Find the equivalent Y circuit for the Δ circuit shown in Figure below.

Solution:
\[ R_A = 90 \ \Omega \]
\[ R_B = 60 \ \Omega \]
\[ R_C = 30 \ \Omega \]

\[ R_1 = \frac{(30\Omega)(60\Omega)}{30\Omega + 60\Omega + 90\Omega} \]

\[ R_1 = \frac{1800\Omega}{180\Omega} = 10\Omega \]
\[ R_2 = \frac{(30\Omega)(90\Omega)}{30\Omega + 60\Omega + 90\Omega} = 15\Omega \]

\[ R_3 = \frac{(60\Omega)(90\Omega)}{30\Omega + 60\Omega + 90\Omega} = 30\Omega \]
**Y-Δ Transformation Equations**

General Idea:

\[ R_\Delta = \frac{\sum (R_{yi}R_{yj}) \text{ all pairs}}{R_{y\text{ opposite}}} \]

\[ R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2} \]

\[ R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3} \]

\[ R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1} \]
EXAMPLE 7. Find the Δ network equivalent of the Y network shown in Figure below.

Solution:

\[ R_1 = 2.4 \text{ kΩ} \]
\[ R_2 = 3.6 \text{ kΩ} \]
\[ R_3 = 4.8 \text{ kΩ} \]
\[ R_A = \frac{(4.8 \, k\Omega)(2.4 \, k\Omega) + (4.8 \, k\Omega)(3.6 \, k\Omega) + (2.4 \, k\Omega)(3.6 \, k\Omega)}{4.8 \, k\Omega} = 7.8 \, k\Omega \]

\[ R_B = \frac{(4.8 \, k\Omega)(2.4 \, k\Omega) + (4.8 \, k\Omega)(3.6 \, k\Omega) + (2.4 \, k\Omega)(3.6 \, k\Omega)}{3.6 \, k\Omega} = 10.4 \, k\Omega \]

\[ R_C = \frac{(4.8 \, k\Omega)(2.4 \, k\Omega) + (4.8 \, k\Omega)(3.6 \, k\Omega) + (2.4 \, k\Omega)(3.6 \, k\Omega)}{2.4 \, k\Omega} = 15.6 \, k\Omega \]
EXAMPLE 8. Find the total resistance of the network of Figure below, where $R_A = 3 \, \Omega$, $R_B = 3 \, \Omega$, and $R_C = 6 \, \Omega$. 
Solution:

\[ R_1 = \frac{R_B R_C}{R_A + R_B + R_C} = \frac{(3 \Omega)(6 \Omega)}{3 \Omega + 3 \Omega + 6 \Omega} = \frac{18 \Omega}{12} = 1.5 \Omega \]

\[ R_2 = \frac{R_C R_A}{R_A + R_B + R_C} = \frac{(6 \Omega)(3 \Omega)}{3 \Omega + 3 \Omega + 6 \Omega} = \frac{18 \Omega}{12} = 1.5 \Omega \]

\[ R_2 = \frac{R_A R_B}{R_A + R_B + R_C} = \frac{(3 \Omega)(3 \Omega)}{3 \Omega + 3 \Omega + 6 \Omega} = \frac{9 \Omega}{12} = 0.75 \Omega \]
Replacing the $\Delta$ by the $Y$, as shown in Figure below, yields

$$R_T = 0.75\Omega + \frac{(4\Omega + 1.5\Omega)(2\Omega + 1.5\Omega)}{(4\Omega + 1.5\Omega) + (2\Omega + 1.5\Omega)}$$

$$R_T = 0.75\Omega + \frac{(5.5\Omega)(3.5\Omega)}{(5.5\Omega) + (3.5\Omega)}$$

$$R_T = 0.75\Omega + 2.139\Omega = 2.889\Omega$$
Thank You!